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A localization of Γ -measurability

V. Gregoriades, N. Papanastassiou *

Department of Mathematics, University of Athens, 15784 Panepistimiopolis, Athens, Greece

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Abstract

We introduce the notion of Γ continuity at a point x —where Γ is any pointclass—and give conditions under which Γ continuity at every x is equivalent to Γ measurability. Using this we extend the notion of the integral of a measurable function. Also we examine the case $\Gamma = \Sigma_\xi^0$, where $(\Sigma_\xi^0)_{\xi < \omega_1}$ is the usual ramification of the class of Borel sets, see [A.S. Kechris, *Classical Descriptive Set Theory*, Springer-Verlag, 1994].

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1. Introduction

With the letters X and Y we always mean metric spaces, unless stated otherwise. Also with the term “pointclass” we mean a collection of sets in arbitrary spaces, for example, the class of open sets. If Γ is a pointclass and X is a metric space we denote with $\Gamma(X)$ the family of the subsets of X which are in Γ .

Recall that if Γ is an arbitrary pointclass and f is a function from X to Y , we call f Γ measurable iff for each open $G \subseteq Y$ the inverse image $f^{-1}[G]$ is also in Γ [1].

We will consider some special cases for Γ . First define the family $\Sigma_1^0(X)$ as the collection of all open subsets of X . Put also $\Pi_1^0(X) = \{A \subseteq X / X \setminus A \in \Sigma_1^0(X)\}$, i.e. the family $\Pi_1^0(X)$ is the collection of closed subsets of X .

By transfinite recursion we define for each $\xi < \omega_1$ [1]

$$\Sigma_\xi^0(X) = \left\{ \bigcup_{n \in \omega} A_n / \text{where } A_n \in \Pi_{\xi_n}^0(X) \text{ for some } \xi_n < \xi, \forall n \in \omega \right\}$$

and

$$\Pi_\xi^0(X) = \{A \subseteq X / X \setminus A \in \Sigma_\xi^0(X)\}.$$

Put also $\Delta_\xi^0(X) = \Sigma_\xi^0(X) \cap \Pi_\xi^0(X)$, for each $\xi < \omega_1$.

* Corresponding author.

E-mail addresses: vasilisg@math.uoa.gr (V. Gregoriades), npapanas@math.uoa.gr (N. Papanastassiou).

It is well known that $\Delta_\xi^0(X) \subseteq \Sigma_\xi^0(X) \subseteq \Delta_{\xi+1}^0(X)$ for each $\xi < \omega_1$ and that $\bigcup_{\xi < \omega_1} \Sigma_\xi^0(X) = \bigcup_{\xi < \omega_1} \Pi_\xi^0(X) = \mathcal{B}(X)$, where $\mathcal{B}(X)$ is the Borel σ -algebra on X .

Recall that a topological space X is called *Polish* iff it is separable and metrizable by some metric d such that (X, d) is complete. If X is a perfect Polish space (i.e. a Polish space with no isolated points), then for each $\eta < \xi < \omega_1$ there exists a set $A \in \Delta_\xi^0(X) \setminus \Delta_\eta^0(X)$. With Σ_ξ^0 we mean the class of all sets which belong to $\Sigma_\xi^0(X)$ for some X .

2. Γ continuity

We now give a notion which is closely connected to Γ measurability.

Definition 2.1. Let Γ be an arbitrary class of sets, a function $f : X \rightarrow Y$ and $x \in X$. The function f is called Γ continuous at x iff for every $\varepsilon > 0$ there exists $\delta > 0$ such that the set $f^{-1}[S(f(x), \varepsilon)] \cap S(x, \delta)$ is in Γ .

Also say that f is Γ continuous iff f is Γ continuous at every $x \in X$.

[Of course with $S(f(x), \varepsilon)$ we mean the set $\{y \in Y / p(f(x), y) < \varepsilon\}$ where p is the metric of Y . The same for $S(x, \delta)$.]

We will denote the set $f^{-1}[S(f(x), \varepsilon)] \cap S(x, \delta)$ with $A(x, \varepsilon, \delta)$ or simpler with $A(\varepsilon, \delta)$.

We first concentrate on the case where $\Gamma = \Sigma_\xi^0$, for some $\xi < \omega_1$. The following are easy consequences of the definitions.

Remark 2.2.

- (I) Assume that Γ contains the class of open sets and is closed under finite intersections. If $f : X \rightarrow Y$ is Γ measurable then f is Γ continuous.
- (II) A function $f : X \rightarrow Y$ is Σ_1^0 continuous exactly when it is continuous, or equivalently iff it is Σ_1^0 measurable.
- (III) Let Σ_ξ^0 measurable functions $f, g : X \rightarrow \mathbb{R}$ and $\lambda \in \mathbb{R}$. Then the functions $f + g, \lambda \cdot f, f \cdot g, |f|, \max\{f, g\}$ and $\min\{f, g\}$ are also Σ_ξ^0 measurable.
- (IV) If $\eta < \xi < \omega_1$ and $f : X \rightarrow Y$ is Σ_η^0 measurable (or Σ_ξ^0 continuous at some $x \in X$), then f is also Σ_ξ^0 measurable (respectively, Σ_ξ^0 continuous at x).

Example 2.3.

- (I) Define $f : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$f(x) = \begin{cases} 0, & \text{if } x \leq 0, \\ 1, & \text{if } x > 0. \end{cases}$$

For each $\varepsilon > 0$ put $\delta = 1 > 0$. Then $A(0, \varepsilon, 1) = (x - 1, 0]$ for each $x \in \mathbb{R}$. Hence $A(0, \varepsilon, 1) \in \Sigma_2^0(\mathbb{R})$. Therefore f is Σ_2^0 continuous at 0, but not continuous at 0.

- (II) Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be the Dirichlet function, i.e.

$$f(x) = \begin{cases} 1, & \text{if } x \in \mathbb{Q}, \\ 0, & \text{if } x \notin \mathbb{Q}. \end{cases}$$

For $x \in \mathbb{Q}$ and $\varepsilon > 0$ the set $A(x, \varepsilon, 1)$ is one of the sets $(x - 1, x + 1) \cap \mathbb{Q}, \mathbb{R}$. In either case $A(x, \varepsilon, 1)$ is an F_σ set and therefore f is Σ_2^0 -continuous on every rational x . Also for each rational x and for each $\delta > 0$ the set $A(x, \frac{1}{2}, \delta) = (x - \delta, x + \delta) \cap \mathbb{Q}$ is not a G_δ set. Hence f is not Π_2^0 -continuous at \mathbb{Q} .

With analogous arguments the Dirichlet function is Π_2^0 but not Σ_2^0 continuous at every $x \notin \mathbb{Q}$.

It is interesting to distinguish Σ_ξ^0 continuity from Σ_η^0 continuity. This is fairly easy to do at this point. However, we will obtain this as a result of Theorem 2.5.

According to Remark 2.2, Σ_ξ^0 measurability implies Σ_ξ^0 continuity. Of course the inverse is also true in the case of $\xi = 1$. This is because the class of Σ_1^0 sets is closed under arbitrary unions. However the latter does not hold

for the general case; so we cannot refer to an analogous proof in order to establish the inverse in the case of an arbitrary ξ .

On the other hand, in some topological spaces we can replace arbitrary unions of open sets with countable ones, i.e. those spaces which satisfy the *Lindelöf property*. It is well known that separable metric spaces are among those spaces. This remark will help us prove that Σ_ξ^0 measurability is equivalent to Σ_ξ^0 continuity in separable metric spaces.

Lemma 2.4. *Let (X, d) and (Y, p) be separable metric spaces and a function $f : X \rightarrow Y$. For each open $G \subseteq Y$ which is bounded (i.e. $\sup_{x,y \in G} p(x, y) < \infty$) there exists a family $(\varepsilon_x)_{x \in f^{-1}[G]}$ of positive reals, such that for every family $(\delta_x)_{x \in f^{-1}[G]}$ of positive reals there exists a countable $I \subseteq f^{-1}[G]$ with*

$$f^{-1}[G] = \bigcup_{x \in I} A(x, \varepsilon_x, \delta_x)$$

(where $A(x, \varepsilon_x, \delta_x) = f^{-1}[S(f(x), \varepsilon_x)] \cap S(x, \delta_x)$).

Roughly speaking this lemma transfers arbitrarily large unions in which we may concern, into countable unions.

Proof. Let $G \subseteq Y$ open and bounded. Then for each $x \in f^{-1}[G]$ there exists $r > 0$ such that $S(f(x), r) \subseteq G$. Put $\varepsilon_x = \sup\{r > 0 / S(f(x), r) \subseteq G\} > 0$. Then $\varepsilon_x \in \mathbb{R}$ since G is bounded.

One can check that $S(f(x), \varepsilon_x) \subseteq G$ for each $x \in f^{-1}[G]$.

Take $(\delta_x)_{x \in f^{-1}[G]}$ any family of positive reals.

Put $\mathcal{A} = \{S(f(x), \frac{\varepsilon_x}{4}) / x \in f^{-1}[G]\}$. From the Lindelöf property of Y there exists a sequence $(x_n)_{n \in \omega}$ of elements of $f^{-1}[G]$ such that $\bigcup_{n \in \omega} S(f(x_n), \frac{\varepsilon_{x_n}}{4}) = \bigcup \mathcal{A}$.

For each $n \in \omega$ put $B_n = \{x \in X / f(x) \in S(f(x_n), \frac{\varepsilon_{x_n}}{4})\} \subseteq f^{-1}[G]$ and $\mathcal{B}_n = \{S(x, \delta_x) / x \in B_n\}$. From the Lindelöf property of X there exists a sequence $(x_k^n)_{k \in \omega}$ of elements of B_n such that $\bigcup_{k \in \omega} S(x_k^n, \delta_{x_k^n}) = \bigcup \mathcal{B}_n$, for each $n \in \omega$.

Define $I = \{x_k^n / n, k \in \omega\}$.

Observe that if $x = x_k^n$ for some $n, k \in \omega$, then from the definition of ε_x it follows that $\varepsilon_{x_n} \leq 2\varepsilon_x$. Using this remark it is easy to verify that

$$f^{-1}[G] = \bigcup_{x \in I} A(x, \varepsilon_x, \delta_x). \quad \square$$

Theorem 2.5. *Let X, Y be separable metric spaces and a function $f : X \rightarrow Y$.*

- (I) *If there exists a function $\xi : X \rightarrow \omega_1$ such that for each $x \in X$ the function f is $\Sigma_{\xi(x)}^0$ continuous at x , then f is Σ_ζ^0 measurable for some countable ordinal $\zeta \leq \sup_{x \in X} \xi(x)$.*
- (II) *For each $\xi < \omega_1$, f is Σ_ξ^0 measurable exactly when it is Σ_ξ^0 continuous at every $x \in X$.*

Proof. It is enough to prove (I) since the second assertion follows immediately from the first.

Let $\{G_n / n \in \omega\}$ be a countable basis for the topology of Y . Since Y satisfies the Lindelöf property we may assume that each G_n is an open ball and hence bounded (with respect to the metric of Y).

For each $n \in \omega$ choose a family of positive reals $(\varepsilon_x^n)_{x \in f^{-1}[G_n]}$ as in Lemma 2.4.

Now fix some $n \in \omega$. For each $x \in f^{-1}[G_n]$, f is $\Sigma_{\xi(x)}^0$ continuous at x . So for the $\varepsilon_x^n > 0$, there exists $\delta_x > 0$ with $A(x, \varepsilon_x^n, \delta_x) \in \Sigma_{\xi(x)}^0$. Choose a family $(\delta_x)_{x \in f^{-1}[G_n]}$ of those δ 's. From Lemma 2.4 there exists a countable set $I \equiv I_n \subseteq f^{-1}[G_n]$ such that

$$f^{-1}[G_n] = \bigcup_{x \in I_n} A(x, \varepsilon_x^n, \delta_x). \quad (*)$$

If we let $\zeta_n = \sup_{x \in I_n} \xi(x) \leq \sup_{x \in X} \xi(x)$ then ζ_n is a countable ordinal and $A(x, \varepsilon_x^n, \delta_x) \in \Sigma_{\zeta_n}^0$ for each $x \in I_n$. So $f^{-1}[G_n] \in \Sigma_{\zeta_n}^0$.

Putting $\zeta = \sup_{n \in \omega} \zeta_n$, where ζ_n is as above, we obtain the result. \square

The preceding theorem allows to view—in separable metric spaces—the notion of Σ_ξ^0 measurability in a “local way”, since Σ_ξ^0 continuity is a local meaning.

This proof applies also to other pointclasses. Let us first recall the class of sets with the *Baire property*. If A is a subset of a topological space X , we say that A has the Baire property iff there exists some open U such that the symmetric difference $(A \setminus U) \cup (U \setminus A)$ is meager in X .

We denote the class of the sets with the Baire property with BP . It is well known that for any topological space X , the family $BP(X)$ forms the least σ -algebra on X which contains all open and all meager sets.

Corollary 2.6. *Let X, Y be separable metric spaces and a function $f : X \rightarrow Y$. Also let a pointclass Γ which contains the open sets and it is closed under countable unions and finite intersections.*

Then f is Γ measurable exactly when it is Γ continuous at every $x \in X$.

In particular f is BP measurable, i.e. Baire measurable (or Σ_ξ^0 measurable) exactly when it is BP continuous at every $x \in X$ (respectively, Σ_ξ^0 continuous at every $x \in X$).

Proof. Proceed as in the proof of Theorem 2.5. \square

We now give an example distinguishing Σ_ξ^0 continuity from Σ_η^0 continuity, for each $\eta < \xi < \omega_1$, using Theorem 2.5.

Example 2.7. Let X be a perfect Polish space, $\eta < \xi < \omega_1$ and A in $\Delta_\xi^0(X)$ but not in $\Delta_\eta^0(X)$.

Then the characteristic function of A , χ_A is easily Σ_ξ^0 measurable (and hence Σ_ξ^0 continuous) but not Σ_η^0 measurable. From the previous theorem there exists some $x \in X$ such that f is not Σ_η^0 continuous at x . Therefore f is Σ_ξ^0 continuous but not Σ_η^0 continuous.

We conclude with an application of Corollary 2.6. Let us begin with some notations. If X is a metric space, A is a subset of X and Γ is a pointclass denote with Γ_A the family $\{G \cap A / G \text{ is a subset of } X \text{ and in } \Gamma\}$. Also for a function $f : X \rightarrow Y$ put $X(f, \Gamma) = \{x \in X / \text{the function } f \text{ is } \Gamma \text{ continuous at } x\}$. Finally for $A \subseteq X$ denote with χ_A the characteristic function of A .

Let X be a separable metric space and a function $f : X \rightarrow \mathbb{R}$. We will consider the case where Γ is a σ -algebra \mathcal{M} which contains the open sets. Also let some $A \subseteq X(f, \mathcal{M}) \equiv X_f$. Then for each $x \in A$ the function f is \mathcal{M} continuous at x . It is clear that the function $f \upharpoonright A$ (i.e. the restriction of f on A), is \mathcal{M}_A continuous at every $x \in A$. Now regard A as a metric space. From Corollary 2.6 it follows that the function $f \upharpoonright A$ is \mathcal{M}_A measurable. If furthermore $A \in \mathcal{M}$ one can verify that the function $f \cdot \chi_A$ is \mathcal{M} measurable.

So if μ is a measure on (X, \mathcal{M}) the integral $\int_A f d\mu$ is well defined for each $A \in \mathcal{M}$ with $A \subseteq X_f$.

Definition 2.8. Let X be a separable metric space, \mathcal{M} be a σ -algebra on X which contains the open sets and μ a measure on (X, \mathcal{M}) . Let also a non-negative function $f : X \rightarrow \mathbb{R}$. Define the *partial integral* of the function f as follows:

$$\int^p f d\mu = \sup \left\{ \int_A f d\mu / A \in \mathcal{M} \text{ \& } A \subseteq X_f \right\}.$$

Remark 2.9. Notice that if X_f is in \mathcal{M} since f is non-negative we have that $\int^p f d\mu = \int_{X_f} f d\mu$. The set X_f is sort of speak the “largest” set on which the function f behaves like a measurable function. It would be interesting to find conditions under which the set X_f is in \mathcal{M} .

If furthermore $X_f = X$ then the partial integral coincides with the usual integral. Thus this new notion is indeed a generalization of the classic one.

Of course we may extend the previous notion to a not necessarily non-negative function with the usual way. For an arbitrary function $f : X \rightarrow \mathbb{R}$ let $\int^p f d\mu = \int^p f^+ d\mu - \int^p f^- d\mu$, where $f^+ = \max\{f, 0\}$ and $f^- = \max\{-f, 0\}$. (In case of $\infty - \infty$ we define the partial integral to be 0.)

It would be interesting also to examine if the classic theorems of the usual integral can be transferred to the partial integral, see [2].

Example 2.10. Here we give an example of a partial integral which is reduced to the Lebesgue integral on a Cantor-type set. Define $x \sim y \Leftrightarrow x - y \in \mathbb{Q}$ and (using the Axiom of Choice) let A be a set which contains exactly one member of each equivalence class. Let \mathcal{M} be the Lebesgue σ -algebra on \mathbb{R} and μ be the Lebesgue measure on $(\mathbb{R}, \mathcal{M})$. It is well known that $A \notin \mathcal{M}$, in fact for every set $U \subseteq \mathbb{R}$, if $U \cap A \in \mathcal{M}$ then $\mu(U \cap A) = 0$. This is because if $\mu(U \cap A) > 0$ then from the Steinhaus theorem there exists $\delta > 0$ such that $(-\delta, \delta) \subseteq (U \cap A) - (U \cap A) \subseteq A - A$, a contradiction.

Let \mathcal{B} be the family of all open intervals $(x - \delta, x + \delta)$ for which $A^c \cap (x - \delta, x + \delta) \in \mathcal{M}$, where A^c stands for the complement of A in \mathbb{R} . If $\mathcal{B} = \emptyset$ put $J = \emptyset$, otherwise choose a countable family $(I_n)_{n \in \omega}$ of the previous intervals which covers \mathcal{B} and put $J = \bigcup_{n \in \omega} I_n$. In any case we have that $A^c \cap J \in \mathcal{M}$ and if $A^c \cap (x - \delta, x + \delta) \in \mathcal{M}$ then $(x - \delta, x + \delta) \subseteq J$.

Define $X = \mathbb{R} \setminus J$. Then X is a closed set and furthermore $\mu(X) > 0$. Otherwise the set $A^c \cap J^c$ would be μ -null and thus in \mathcal{M} . Hence $A^c = (A^c \cap J) \cup (A^c \cap J^c) \in \mathcal{M}$, a contradiction. It is well known that we can find a Cantor type set $C \subseteq X$ for which $\mu(C) > 0$.

Let \mathcal{M}_X be the restriction of \mathcal{M} on X , i.e. $\mathcal{M}_X = \{M \cap X / M \in \mathcal{M}\}$. Also let μ_X be the restriction of μ on \mathcal{M}_X . Since X is closed and thus in \mathcal{M} it is clear that some $B \subseteq X$ is in \mathcal{M}_X if and only if $B \in \mathcal{M}$.

Define the function $g : X \rightarrow \mathbb{R}$ by $g(x) = \inf\{|x - y| / y \in C\}$. The function f is continuous non-negative and $g^{-1}[\{0\}] = C$.

Now define $f : X \rightarrow \mathbb{R}$ as follows

$$f(x) = \begin{cases} g(x) + 1, & \text{if } x \in A, \\ 1, & \text{if } x \notin A. \end{cases}$$

If $x \in C$, i.e. $g(x) + 1 = 1$ it is clear that f is continuous (and hence \mathcal{M}_X continuous) at x . For simplicity put $X_f = X(f, \mathcal{M}_X)$. Let now some $x \in X_f \setminus C$. Assume furthermore that $x \in A$. Then $g(x) > 0$ and $f(x) = g(x) + 1$. Put $\varepsilon = \frac{g(x)}{2}$. Notice that if $y \notin A$ then $f(y) = 1 < g(x) + 1 - \varepsilon = f(x) - \varepsilon$. Hence if $f(y) \in (f(x) - \varepsilon, f(x) + \varepsilon)$ then $y \in A$. It follows that

$$\begin{aligned} f^{-1}[(f(x) - \varepsilon, f(x) + \varepsilon)] &= f^{-1}[(f(x) - \varepsilon, f(x) + \varepsilon)] \cap A \\ &= h^{-1}[(h(x) - \varepsilon, h(x) + \varepsilon)] \cap A, \end{aligned}$$

where $h = g + 1$.

Since $x \in X_f$ there exists some $\delta > 0$ such that $f^{-1}[(f(x) - \varepsilon, f(x) + \varepsilon)] \cap (x - \delta, x + \delta) \in \mathcal{M}_X \subseteq \mathcal{M}$. Also we have that

$$\begin{aligned} f^{-1}[(f(x) - \varepsilon, f(x) + \varepsilon)] \cap (x - \delta, x + \delta) &= h^{-1}[(h(x) - \varepsilon, h(x) + \varepsilon)] \cap A \cap (x - \delta, x + \delta) \\ &= U_x \cap A, \end{aligned}$$

where U_x is open in X such that $x \in U_x$. Therefore $U_x \cap A \in \mathcal{M}_X \subseteq \mathcal{M}$ and thus $\mu(U_x \cap A) = 0$. Repeat the same procedure for each $x \in (X_f \setminus C) \cap A$ in order to get the previous set U_x . Using the Lindelöf property of X we find a sequence $(x_n)_{n \in \omega}$ in $(X_f \setminus C) \cap A$ such that $(X_f \setminus C) \cap A = \bigcup_{n \in \omega} U_{x_n} \cap A$. It follows that the set $(X_f \setminus C) \cap A$ is μ -null and thus it belongs in \mathcal{M} .

Now towards a contradiction assume that there exists some $x \in X_f \setminus C$ which is not in A . Then $f(x) = 1$ and $g(x) > 0$. Put $\varepsilon = \frac{g(x)}{2} > 0$. Since g is continuous there exists some δ_0 such that for all $y \in (x - \delta_0, x + \delta_0) \cap X$ we have that $g(y) > \frac{g(x)}{2}$. Also $x \in X_f$ hence there exists some $\delta > 0$ such that $f^{-1}[(f(x) - \varepsilon, f(x) + \varepsilon)] \cap (x - \delta, x + \delta) \in \mathcal{M}_X \subseteq \mathcal{M}$. We may assume that $\delta \leq \delta_0$. Thus if $y \in (x - \delta, x + \delta) \cap X$ then $f(y) \in (f(x) - \varepsilon, f(x) + \varepsilon) \Leftrightarrow y \notin A$. It follows that $f^{-1}[(f(x) - \varepsilon, f(x) + \varepsilon)] \cap (x - \delta, x + \delta) = A^c \cap (x - \delta, x + \delta) \cap X = A^c \cap J^c \cap (x - \delta, x + \delta) \in \mathcal{M}$.

Since $A^c \cap J \in \mathcal{M}$ it follows that $A^c \cap J \cap (x - \delta, x + \delta) \in \mathcal{M}$ as well. Therefore $A^c \cap (x - \delta, x + \delta) = [A^c \cap J^c \cap (x - \delta, x + \delta)] \cup [A^c \cap J \cap (x - \delta, x + \delta)] \in \mathcal{M}$. From the definition of J it follows that $(x - \delta, x + \delta) \subseteq J$. This is a contradiction since $x \in X = \mathbb{R} \setminus J$.

Therefore $X_f \setminus C = (X_f \setminus C) \cap A$ is μ -null and thus in \mathcal{M} . Since $X_f \setminus C \subseteq X$ we have that $X_f \setminus C \in \mathcal{M}_X$. Furthermore C is closed in X and it is also a subset of X_f , hence $X_f \in \mathcal{M}_X$.

Thus $\int^P f \, d\mu_X = \int_{X_f} f \, d\mu_X = \int_C f \, d\mu_X = \int_C 1 \, d\mu_X = \mu_X(C) = \mu(C) > 0$.

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